

Graphing Techniques

ISI & CMI Yearlong Course 2024

1 What is a graph?

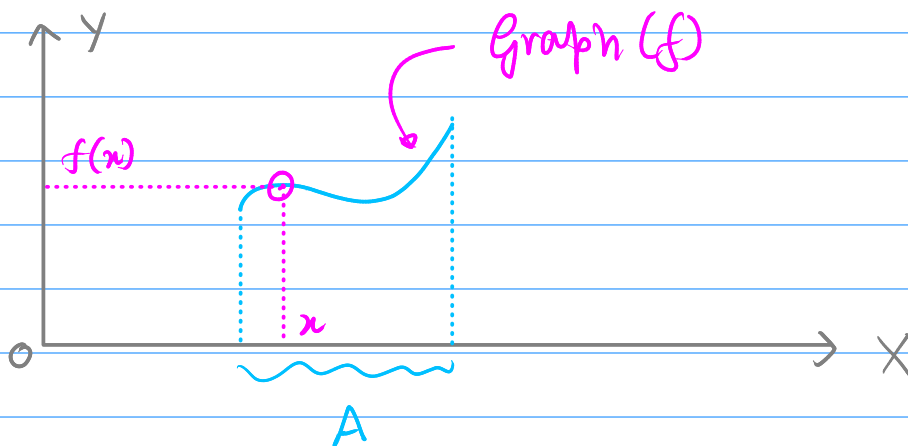
- Suppose we have $f: A \rightarrow \mathbb{R}$, $A \subseteq \mathbb{R}$.

$$\text{Graph}(f) := \{ (\underbrace{x}_{\text{abscissa}}, \underbrace{f(x)}_{\text{ordinate}}) \mid \underbrace{x \in A}_{\text{domain}} \} \subseteq \mathbb{R}^2$$

In other words, the graph of a f^n is the set of points it passes through.

- Sketch of a graph

We usually draw graphs on a plane. ($\mathbb{R}^2$)



Think: $f: \mathbb{D} \rightarrow \mathbb{D}$. How to sketch this graph?

$$\hookrightarrow \text{Graph}(f) = \{ (x, f(x)) \mid \dots \} \subseteq \mathbb{D}^2 \cong \mathbb{R}^2$$

Here on, we will concern ourselves with sketches of graphs of real-valued f^n s only.

2

Why graphs?

- While solving problems, what do we want in essence?
 → Some information about the behaviour of the f^n (continuity, roots, asymptotes, etc--)
- A graph gives us a LOT of information about the behaviour of a f^n in a easy-to-read manner. Like:
 - i) Roots
 - ii) Asymptotes
 - iii) Discontinuities
 - iv) Cusps (Sharp corners)
 - v) Increasing/decreasing
 - vi) Concave / convex
 - vii) Roughly compare growth rate, curvature, etc.
- In fact, knowing the full graph of a f^n means that we know its output at every point in the domain. $\therefore$ We know everything about the f^n .
 Hence, being able to draw a good rough sketch for a graph is a powerful skill!
- Warning:
 Drawing a graph & observing based on that doesn't constitute a proof from exam POV.
 $\therefore$ Use freely in objective qns only.

3 Polynomials

- $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$
 $\dots$ is a degree 'n' polynomial. ($a_n \neq 0$)
- $p(x) = (x-b_1)^{m_1} (x-b_2)^{m_2} \dots (x-b_k)^{m_k} \phi(x)$
 $\dots$ where $b_1, \dots, b_k$ are distinct real roots of $p(x)$, and $m_1, \dots, m_k$ are their respective (algebraic) multiplicities.
 $\& \phi(x)$ has no real roots.

(for now, assume p has at least one real root)

- $$\begin{cases} p(x) \rightarrow \text{sgn}(a_n) \cdot \infty & \text{as } x \rightarrow +\infty \\ p(x) \rightarrow \text{sgn}(a_n) \cdot (-1)^n \cdot \infty & \text{as } x \rightarrow -\infty \end{cases}$$

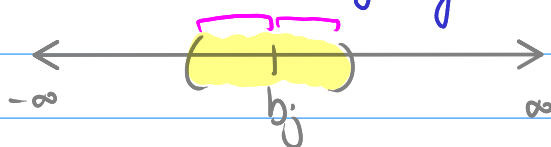
$$\therefore p(x) \sim a_n x^n \text{ as } |x| \rightarrow \infty$$

$$p(0) = a_0 \quad \parallel \quad \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ +1, & x > 0 \end{cases}$$

- Polynomials are continuous, hence have IVP.
 $\therefore$ Sign change can only occur at roots.

(for now) IVP = can draw graph without lifting pen.
 $\rightarrow$ devoid of other roots.

In the vicinity of some root, say b_j ...



$\bigcirc \in A$ open $\rightarrow$ imposes some technical constraints on A .

$$(x-b_1)^{m_1} \dots (x-b_{j-1})^{m_{j-1}} \boxed{(x-b_j)^{m_j}} \dots \phi(x)$$

$\underbrace{\hspace{10em}}$ in locality, fixed sign.

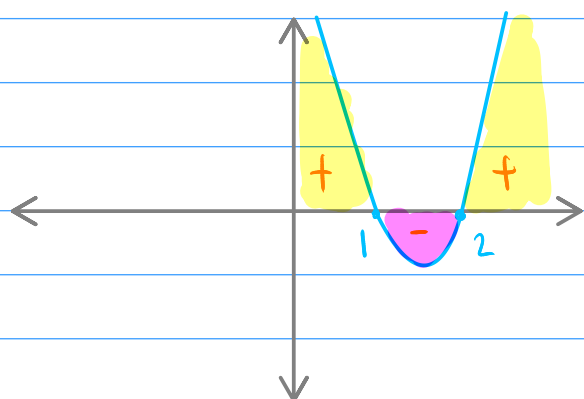
sign change governed by this term.

$\underbrace{\hspace{10em}}$ no real roots, so fixed sign.

$\therefore$ There is a sign change @ b_j iff m_j is odd.

• This is called the "wavy-curve" method.

• Example: $f(x) = x^2 - 3x + 2 = (x-1)(x-2)$



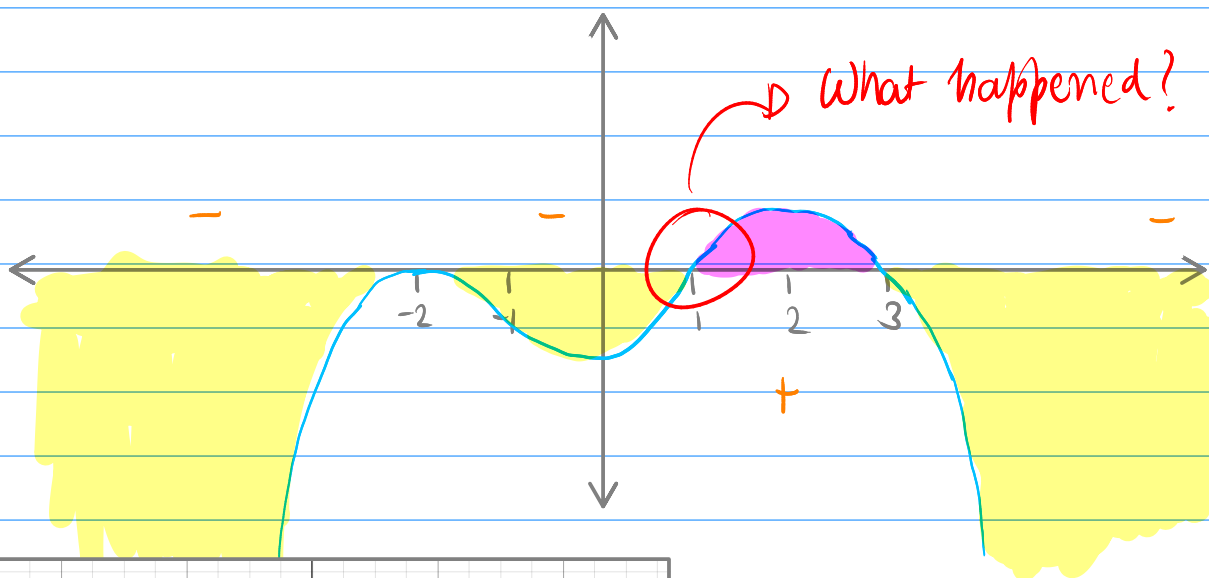
$$f(+\infty) = +\infty$$

$$f(-\infty) = +\infty$$

$$f(0) > 0$$

...

• Example: $f(x) = -3(x+2)^2(x-1)^3(x-3)$

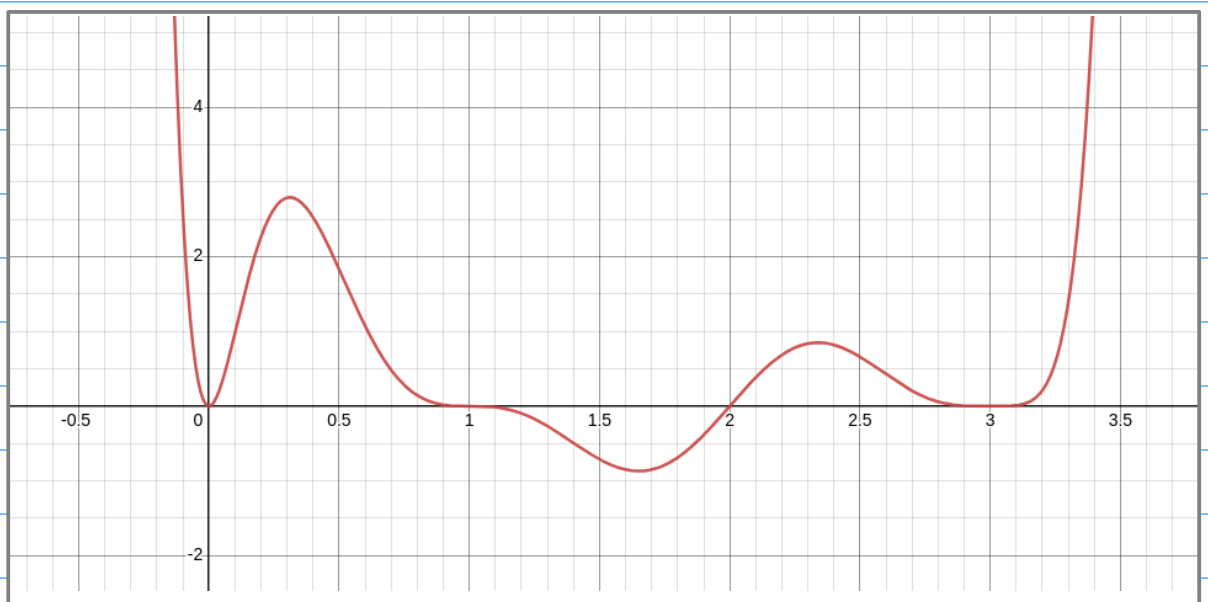


← Actual graph (desmos)

very cube-ish.

- Takeaway: Around b_j , the graph of p , resembles that of $\pm(x-b_j)^{m_j}$ based on other terms
 $\Rightarrow$ Except the sign, we can ignore the effect of other terms!

- Example: $f(x) = x^2(x-1)^3(x-2)(x-3)^4$



Actual

- $p^{(-j)}(x) = (x-b_1)^{m_1} \cdots (x-b_{j-1})^{m_{j-1}} \{ (x-b_{j+1})^{m_{j+1}} \cdots \phi(x) \}$
 $p^{(-j)}(b_j)$ gives you an idea of steepness.

- What if p has no real roots?

Eg.: $p(x) = x^2 + 1$.

Simply graph $p'(x) = p(x) - c$ (for some appr. $c \in \mathbb{R}$)

Then shift whole graph upwards by c .

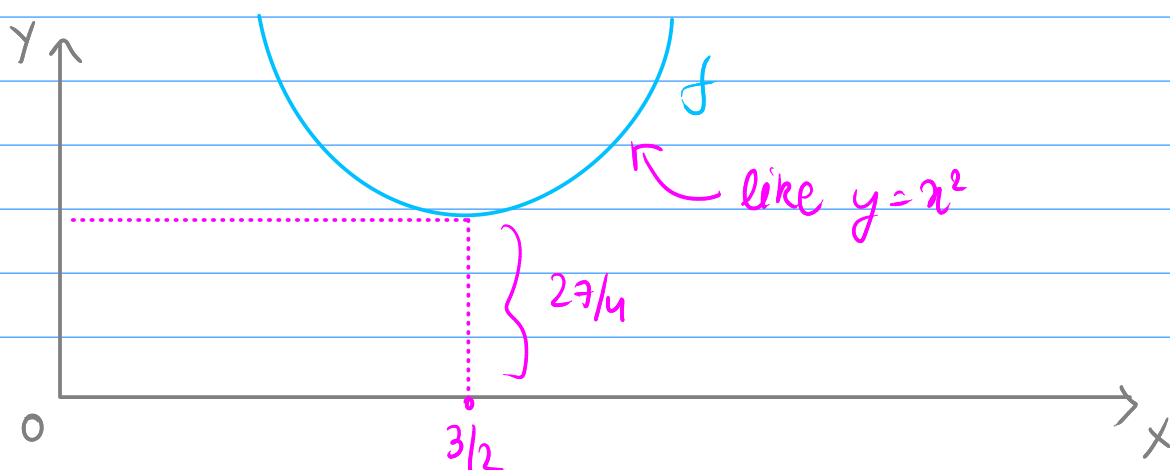
Basically, graph x^2 , shift up by 1.



⑨ $f(x) = x^2 - 3x + 9$ (HW)

(Solution : 2023-08-29)

$$\begin{aligned} f(x) &= x(x-3) + 9 \\ &= (x - 3/2)^2 + 27/4 \\ &= (x-1)(x-2) + 7 \end{aligned} \quad \checkmark$$



4

Rational Functions

- Rational functions $r(x) = \frac{p(x)}{q(x)}$, where p, q are polynomials.

- $$r: \mathbb{R} \setminus [q=0] \rightarrow \mathbb{R}$$

$$x \mapsto \frac{p(x)}{q(x)}$$

$$p(x) = a_0 + a_1x + \dots + a_nx^n = (x-b_1)^{m_{p,1}} \dots (x-b_k)^{m_{p,k}} \phi(x)$$

$$q(x) = c_0 + c_1x + \dots + c_{n_q}x^{n_q} = (x-d_1)^{m_{q,1}} \dots (x-d_l)^{m_{q,l}} \psi(x)$$

where $a_{n_p} \neq 0$, $a_{n_q} \neq 0$, $\{\underbrace{b_1, \dots, b_k}_{\text{roots}}, \underbrace{d_1, \dots, d_l}_{\text{holes}}\}$ are distinct
 & ϕ, ψ have no real roots.

$\sim \frac{1}{x}$: dis contin asymptote

- If $n_p < n_q$:

$$r(x) \rightarrow \text{sgn}\left(\frac{a_{n_p}}{c_{n_q}}\right) \cdot 0 \quad \text{as } x \rightarrow \infty$$

$$r(x) \rightarrow \text{sgn}\left(\frac{a_{n_p}}{c_{n_q}}\right) \cdot (-1)^{n_q - n_p} \cdot 0 \quad \text{as } x \rightarrow -\infty$$

- If $n_p > n_q$:

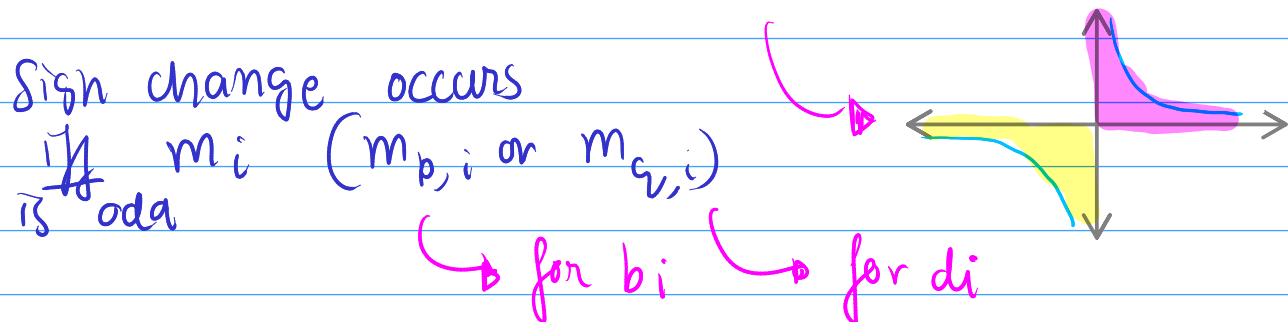
$$r(x) \rightarrow \text{sgn}\left(\frac{a_{n_p}}{c_{n_q}}\right) \cdot \infty \quad \text{as } x \rightarrow \infty$$

$$r(x) \rightarrow \text{sgn}\left(\frac{a_{n_p}}{c_{n_q}}\right) \cdot (-1)^{n_p - n_q} \cdot \infty \quad \text{as } x \rightarrow -\infty$$

- If $n_p = n_q$: $r(x) \rightarrow \frac{a_{n_p}}{c_{n_q}}$

- Sign change can occur @ roots (by crossing x -axis) or @ holes (through asymptotes)

Sign change occurs
if m_i is odd

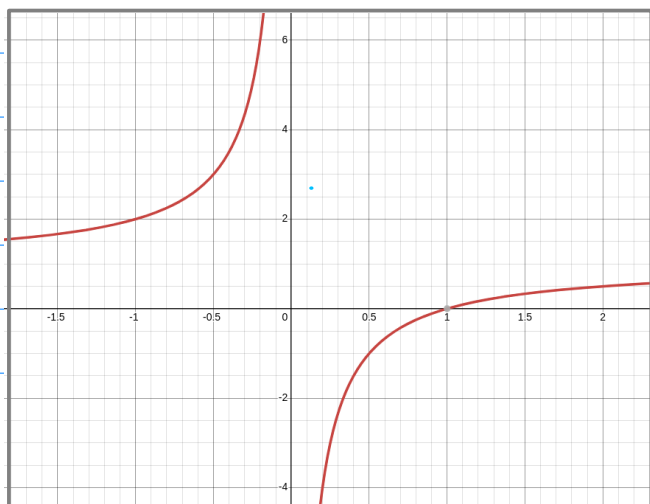
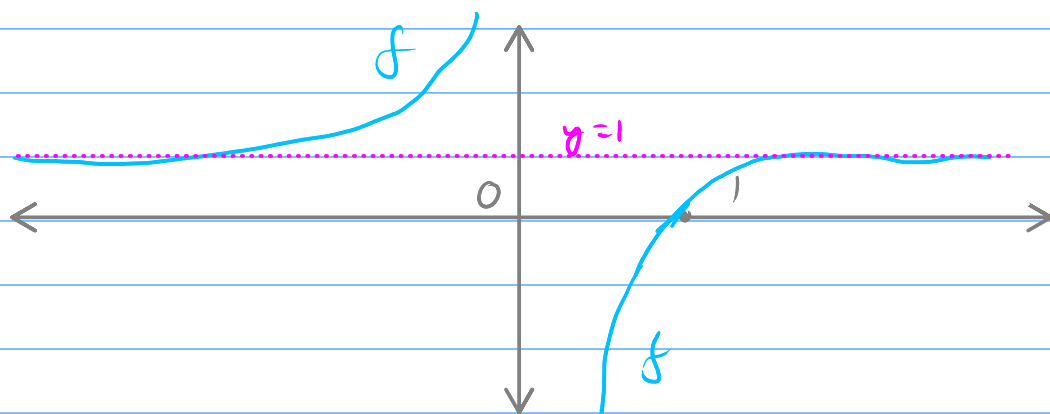


for b_i for d_i

- g exerts the reverse of its local effect, p

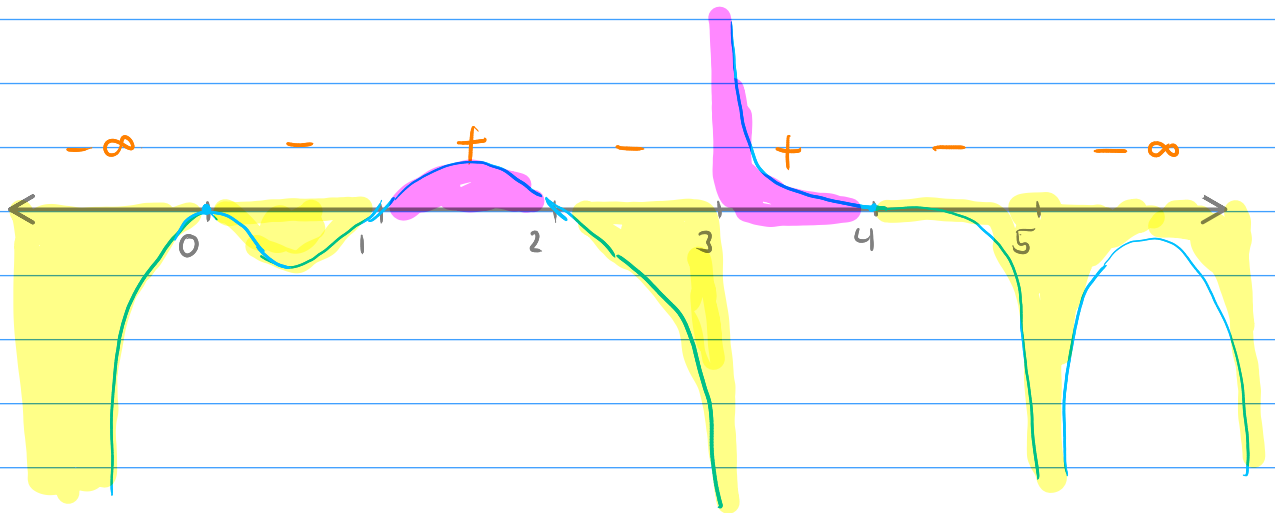
↳ You can still ignore, but for g , instead of resembling, you get the opposite effect.

- Example $f(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$

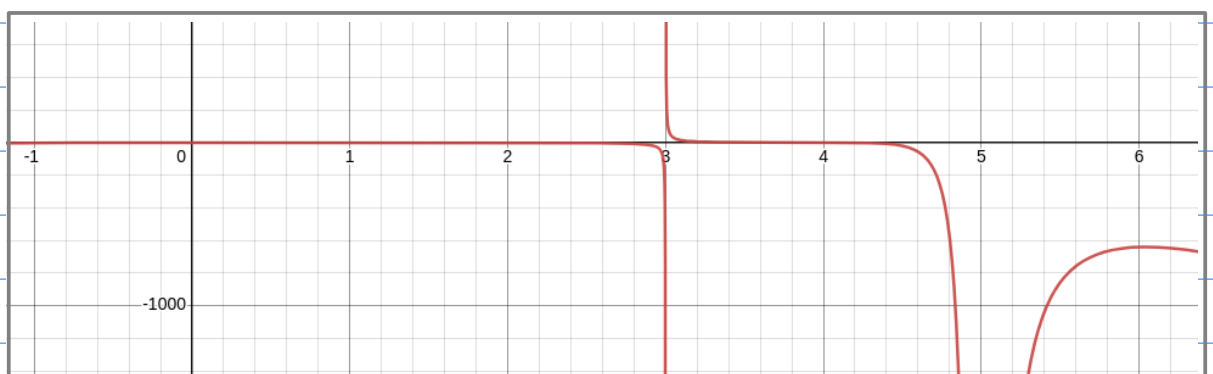


← Actual graph from Desmos.

• Example $f(x) = \frac{x^2(x-1)(x-4)^3(2-x)}{3(x-3)(x-5)^2}$



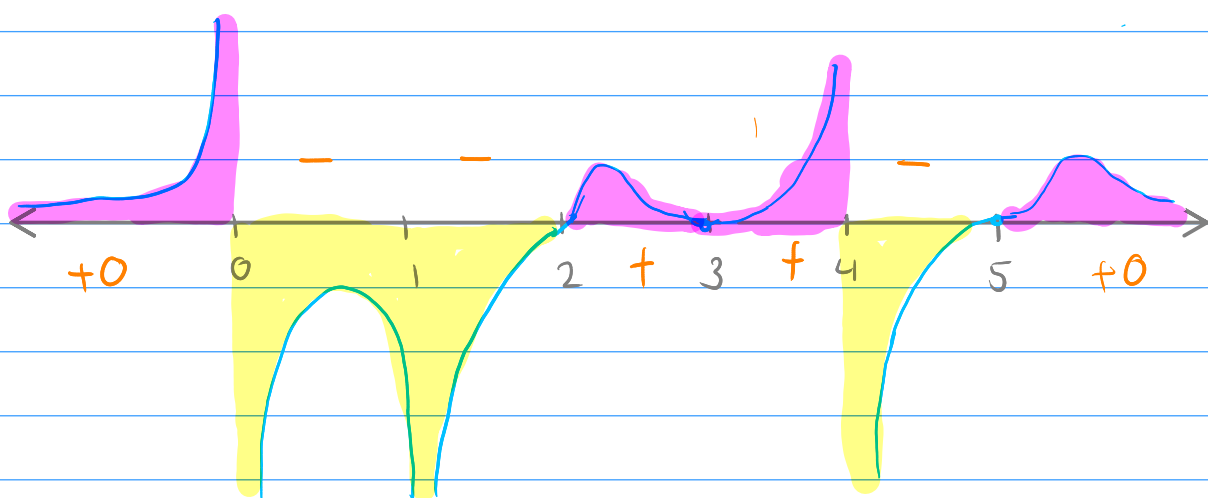
Actual



Q $f(x) = \frac{(x-2)(x-3)^2(x-5)^3}{x^3(x-1)^4(x-4)}$ (HW)

(Solution: 2023-08-29)

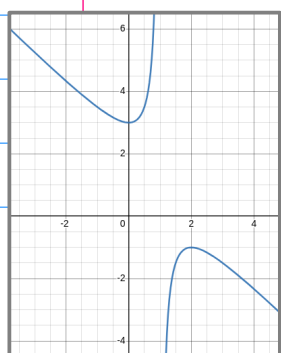
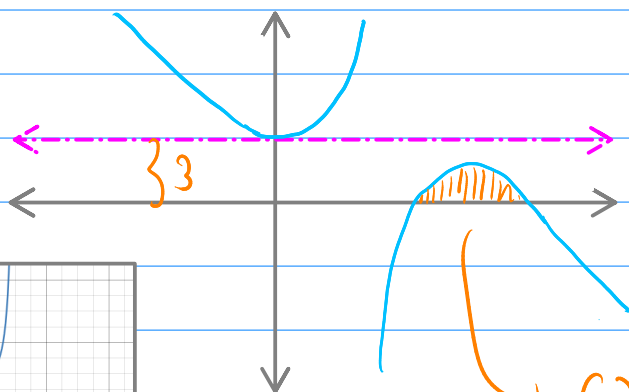
[Check the actual graph yourself]



(not to scale)

= 2023-08-29: (#2) - Graph Transformations.

Q $f(x) = \frac{x^2}{1-x} + 3 = \frac{x^2 - 3x + 3}{1-x}$



Actual

$$= \frac{(x-1)(x-2) + 1}{1-x}$$

$$= 2 - x + \frac{1}{1-x}$$

keep in mind.

Q) Something above 0? No

Try putting $x = 2$ why?

$$= \frac{4}{1-2} + 3 = -1$$

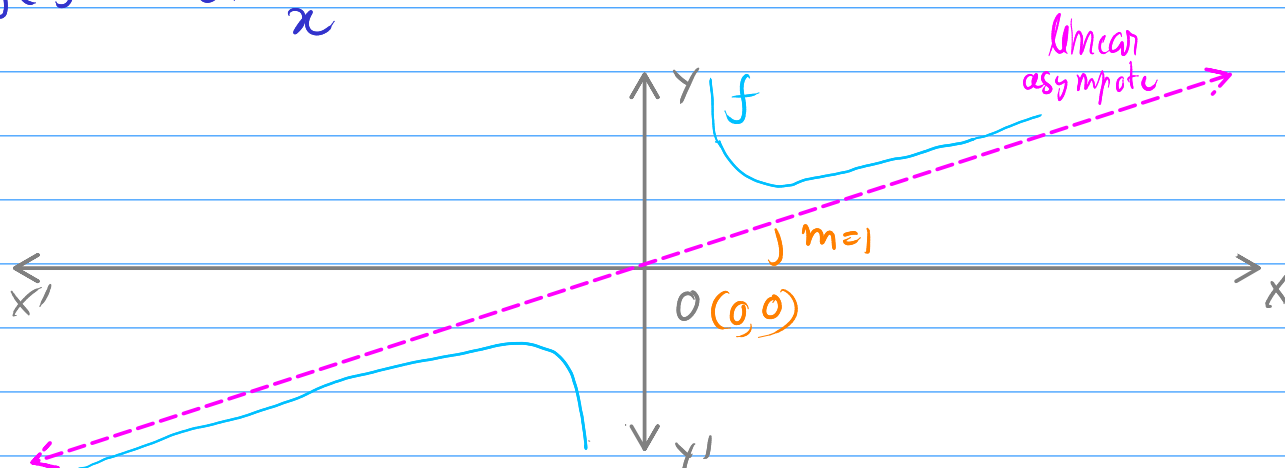
5 Some Standard Graphs

- By now, we know how to graph: (mostly)

Revise before proceeding.

- i) Polynomials, rationals v) x^p , $p \in \mathbb{R}$
 - ii) Logarithms, exponentials
 - iii) Trig functions & inverse trig functions
 - iv) $\lfloor x \rfloor$, $\lceil x \rceil$, $\{x\}$, $|x|$, $\text{sgn}(x)$, $\mathbb{1}_A$; $A \subseteq \mathbb{R}$
- $\downarrow$ floor $\downarrow$ ceil $\downarrow$ frac $\downarrow$ abs $\downarrow$ signum $\downarrow$
- $\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$

- $f(x) = x + \frac{1}{x}$



Near $x=0$, $x \approx 0$, so $f(x) \approx \frac{1}{x}$

For from $x=0$, $\frac{1}{x} \approx 0$, so $f(x) \approx x$

Think: $f(x) = \sin(x) + \frac{1}{x}$

What will be the asymptotic behaviour?
 $|x| \rightarrow \infty \Rightarrow \frac{1}{x} \rightarrow 0$

$\therefore f$ behaves like $\sin(x)$.

6 Graph Transformations

o Translation

- Suppose you know the graph of $f(x)$.
What will be the graph of $g(x) := f(x) + c, c \in \mathbb{R}$?

Simply shift up by c . (y-translation ✓)

- Suppose you know the graph of $f(x)$.
What will be the graph of $g(x) = f(x-a), a \in \mathbb{R}$?

Claim: Shift right by a → why?

Visualize $\Rightarrow g(0) = f(-a) ; g(a) = f(0)$

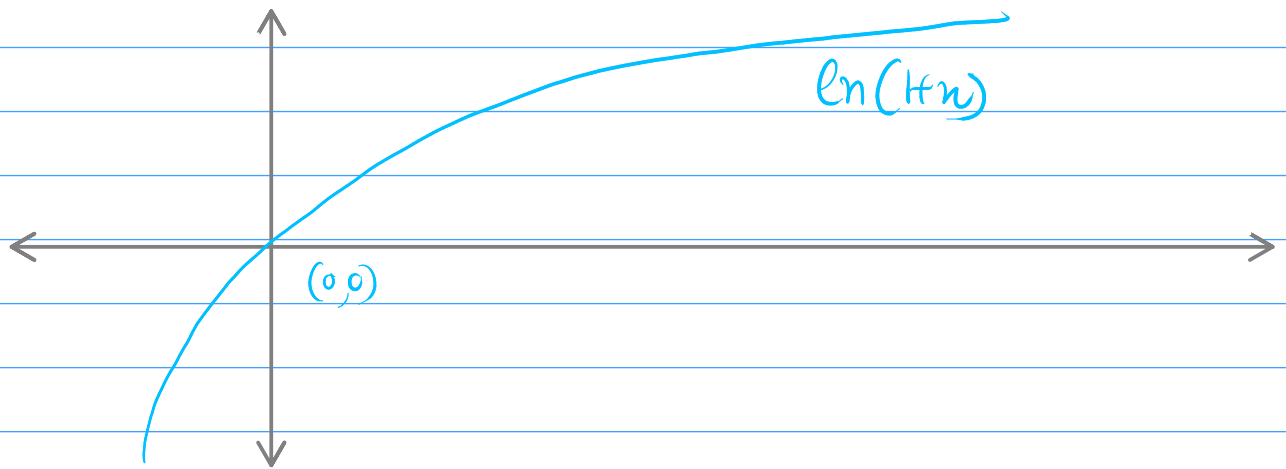
$$\begin{aligned}\text{Graph}(g) &= \{ (x, g(x)) \mid x \in \text{Dom}(g) \} \\ &= \{ (x, f(x-a)) \mid x \in \text{Dom}(f) + a \} \\ &\quad \text{right shift!}\end{aligned}$$

Now $\forall x \in \text{Dom}(f) + a$

$\exists! x' \mid x = x' + a$, where $x' \in \text{Dom}(f)$

$$\begin{aligned}\text{Graph}(g) &= \dots = \{ (\underbrace{x'+a}_{\text{right shift}}, f(x')) \mid x' \in \text{Dom}(f) \} \\ &= \text{Graph}(f) + \underbrace{(a, 0)}_{\text{right shift.}}\end{aligned}$$

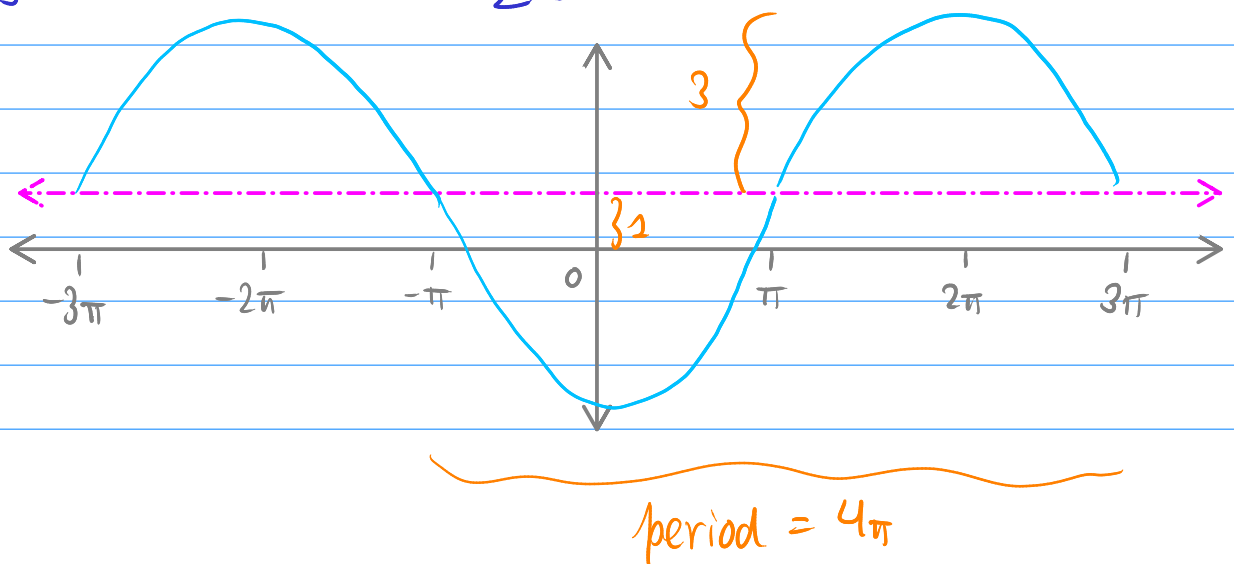
- Example: $f(x) = \ln(1+x)$



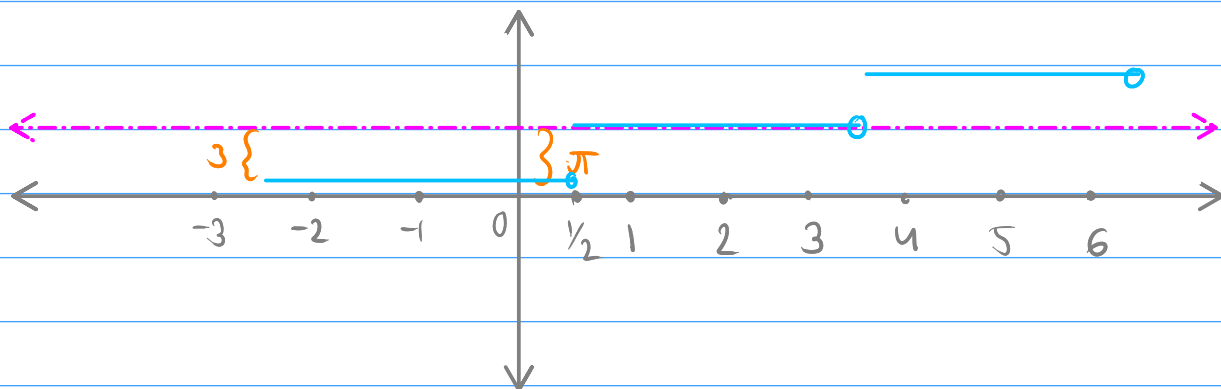
⊙ Scaling

- $g(x) = c f(x)$. Simply stretch by factor of c along y -axis, flipping if $c < 0$.
 - $g(x) = f(ax)$. Simply shrink by factor of a along x -axis, flipping if $a < 0$.
- Proof: Exc

⑥ $f(x) = 3 \sin\left(\frac{x - \pi}{2}\right) + 1$



Q $f(x) = 3 \left\lfloor \frac{x - 1/2}{3} \right\rfloor + \pi$



⑥ Addition

- $f(x) = f_1(x) + f_2(x)$. How to graph $f(x)$!
we know their graphs

$\Rightarrow$ Add the values.

- Same sign $\rightarrow$ reinforce
 Opposite sign $\rightarrow$ reduce $\Leftarrow$ compare relative effects
 Same for monotonicity, convexity, etc.

- Example: $f(x) = x^2 + \ln(x)$

Both are increasing, x^2 is convex, $\ln(x)$ is concave

- Effect Strength (@ $\pm\infty$) $\sin, \cos \rightarrow O(1)$
 $\rightarrow O(e^{\text{poly}(n)})$

$O(x^n) > \{ O(e^n) \gg \text{Polynomials} > \log(x) > \log^{(i)}(x) \gg \log^{(i+1)}(x) \geq O(1)$
 $\hookrightarrow n!$ $\hookrightarrow 2^n$ based on degree (also include $\sqrt{n}$ etc)

all rationals w/ $n_q > n_p$
Also include $\frac{1}{\sqrt{n}}$ etc.

$$O(1) > O\left(\frac{1}{\log x}\right) >> O\left(\frac{1}{\text{Poly}}\right) >> O(e^{-n})$$

based on degree 4

$$(e^{-1/4 \text{ Poly}(n)}) \gtrsim \emptyset$$

$>> O(x^{-n})$

- Effect Strength (near vertical asymptote)

$$O(e^{1/4 \text{ Poly}(n)}) >> \frac{1}{\text{Poly}(n)} > \log(n) > \dots$$

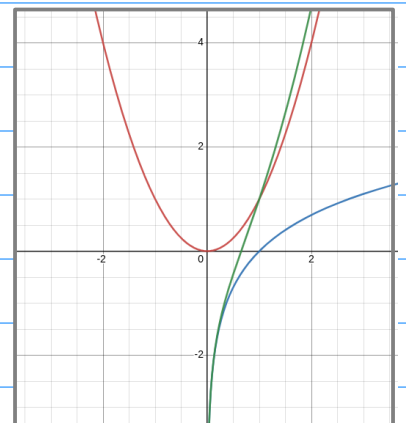
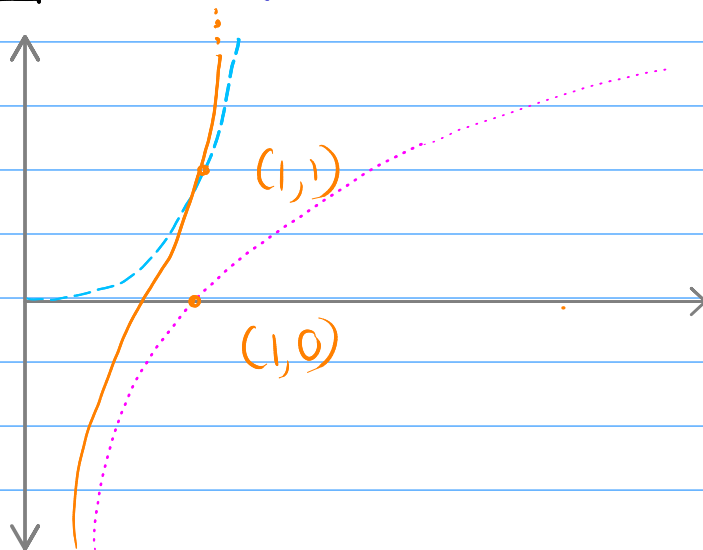
$\hookrightarrow e^{1/4 n^3}$

$e^{-1/n} \sim \sec, \csc, \tan, \cot$

rationals: $n_q > n_p$

ignore,

- Example $f(n) = n^2 + \ln(n)$



Actual

- ① Multiplication (works very similar to addition)

$\Rightarrow$ study in isolation

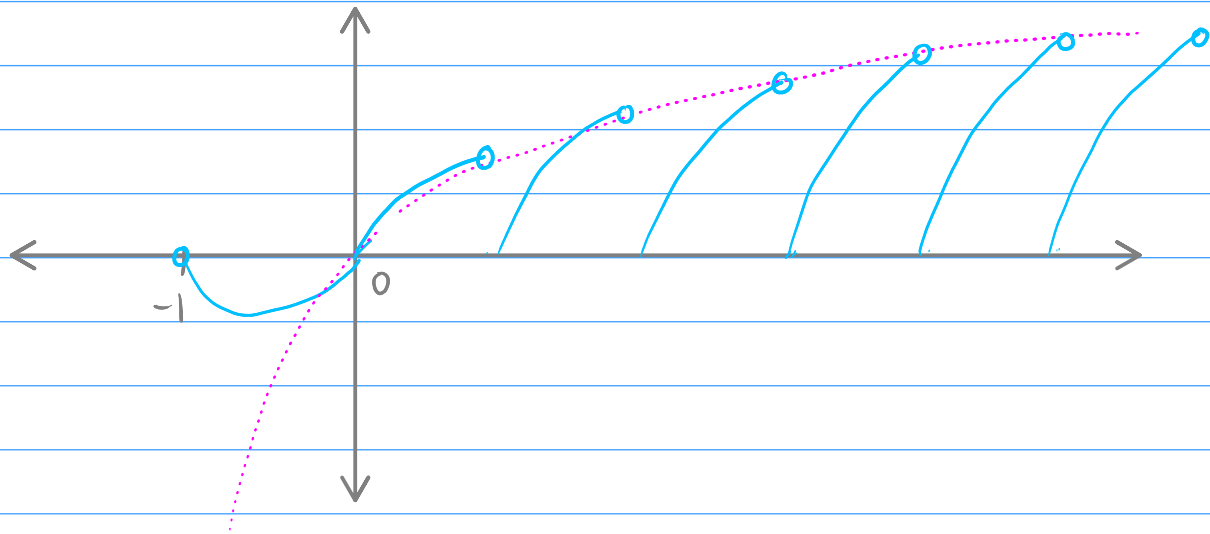
⑥ i) $f(x) = \{x\} \ln(1+x)$

ii) $f(x) = \sin(x) e^{\lfloor \frac{x}{2\pi} \rfloor}$

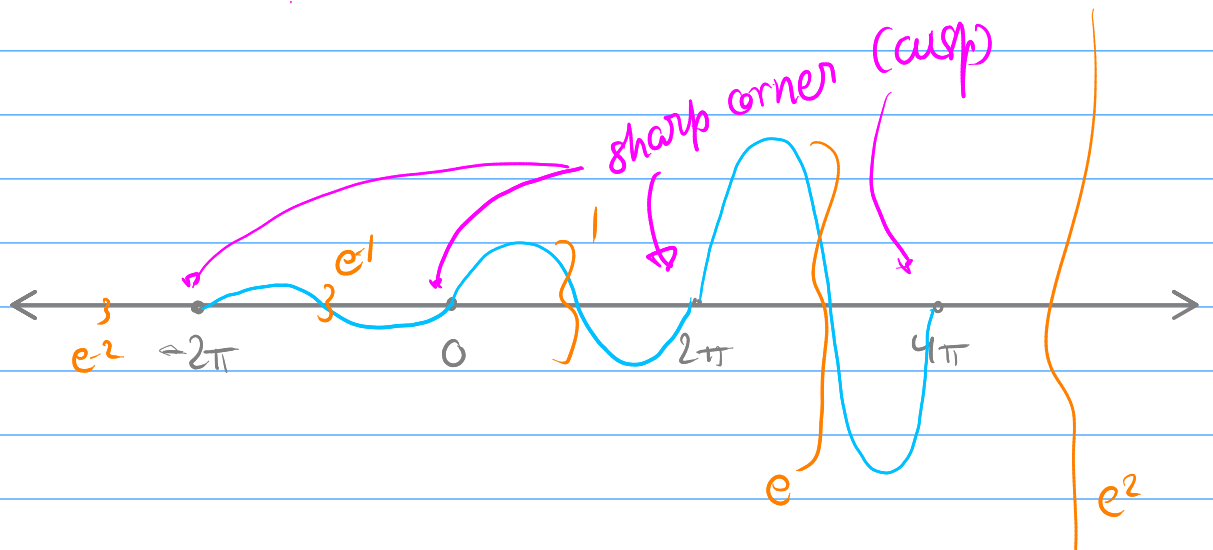
iii) $f(x) = e^{-x} |\ln(x)|$

Ans

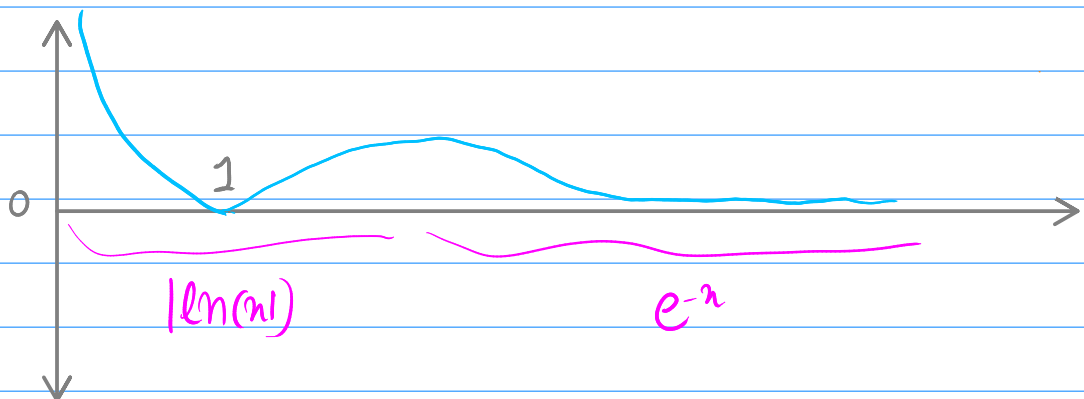
i)



ii)



iii)



⊙ Inversions

$$(x, y) \rightarrow y = f(x)$$

- Suppose you know the graph of f . How to find graph of $g := f^{-1}$? $\rightarrow (x, y) \rightarrow x = f(y)$

Switch abscissa & ordinate

Simply switch!

- Example: e^x & $\ln(x)$

(HW)

(take mirror image wrt line $y = x$)

⊙ Compositions

- Suppose you can sketch g, f . How to sketch $h := g \circ f$?

$\Rightarrow$ No general answer

- If g is piecewise, discrete $\rightarrow [a], \mathbb{I}_\mathbb{A}, \text{sgn}$

$\hookrightarrow$ Then evaluate exhaustively for ranges.

- If g is piecewise, periodic $\rightarrow \{x\}$

$\hookrightarrow$ Restrict domain to one period only & repeat. $\rightarrow |x|$

- If g is piecewise but neither discrete nor periodic

$\hookrightarrow$ Break into cases

- If g is smooth, periodic (trig fns)
 - ↳ Restrict domain to one period only. & repeat
- If $g \sim xp$ (possibly restricts domain $\rightarrow \sqrt{x}$)
 - ↳ Some sort of progressive distortion
- If $g \sim \exp$
 - ↳ $0 \mapsto 1, -\infty \mapsto 0, \infty \mapsto \infty$ faster.
- If $g \sim \ln$ (added domain restrictions!)
 - ↳ $1 \mapsto 0, 0 \xrightarrow{\text{fast}} -\infty, \infty \mapsto \infty$ slower
- If $g \sim \frac{1}{x}$ (added domain restrictions!)
 - ↳ Opposite effect.
- If $g \sim$ complicated poly Hopeless

In many scenarios, graphing too difficult.

②
$$f(x) = \frac{x^2}{1-x} + 3 = \dots = 2 - x + \frac{1}{1-x}$$

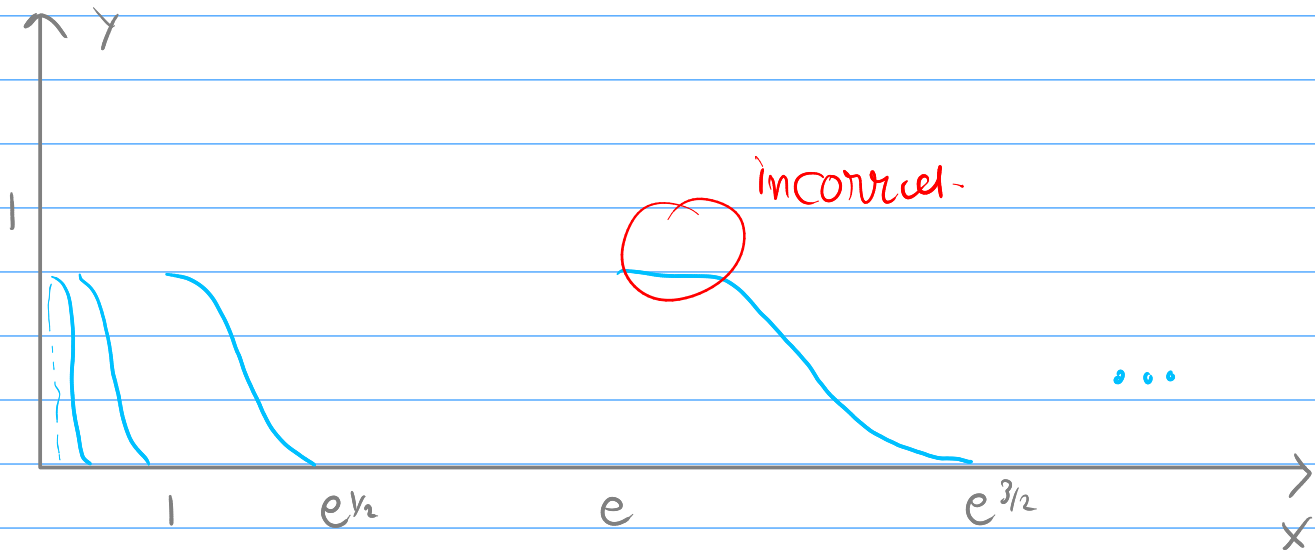
$$= 1 + \left\{ (1-x) + \frac{1}{(1-x)} \right\}$$

How: Graph using translation
& Scaling. (use $x + \frac{1}{x}$ graph)

- (a) i) $f(n) = \lfloor \ln n \rfloor$ & compare the two.
 ii) $g(n) = \ln \lfloor n \rfloor$



(a) $f(x) = e^{-\sqrt{\tan(\pi \ln(x))}}$



Actual.

7

Infinite Oscillations

- We will use $\sin(\frac{1}{x})$, $\cos(\frac{1}{x})$, --- class of functions to generate infinite density oscillations near a certain point.
- See Also: $x^p \sin(\frac{1}{x^q})$ $0 \leq p \leq 3$, $q=2,3$
 $\hookrightarrow$ with revisits in calculus



$\sin(1/x)$

- These fns are interesting because they are ^{intended.} bounded but can be made C^k for any $k \in [0, \infty]$ $\downarrow$
 Useful in calculus, & examiners' favourite!

// END